Frequency Domain Representations of Sampled and Wrapped Signals

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1 Introduction

These notes examine the relationships between frequency domain representations of discrete-time and wrapped signals derived from a continuous-time signal. The first part of these notes develops the relationships for periodic signals which allow for the analysis of periodic signals within the framework of the Fourier transform.

The second part examines the relationships between the Fourier series, the Discrete-Time Fourier Transform (DTFT) and the Discrete Fourier Transform (DFT).

Throughout this document, round brackets are used for functions of continuous variables (examples: $v(t)$ and $V(\omega)$); square brackets are used for functions of discrete variables (example: $v[n]$). In the first part of this document, the equations shown within boxes are results that are used in the developments leading to formulations for the Fourier transform of periodic signals. In the second part of this document, the equations shown within boxes are results which appear on the diagram relating the Fourier domain representations of sampled and wrapped signals.

2 Continuous-Time Fourier Transform

The Fourier transform of a continuous-time signal is given by

$$V(F) = \int_{-\infty}^{\infty} v(t) e^{-j2\pi Ft}\, dt. \tag{1}$$

This is well-defined (converges) if $v(t)$ satisfies the Dirichlet conditions (absolute integrability, finite number of finite discontinuities in a finite time interval, finite number of extrema in a finite interval). The inverse transform is

$$v(t) = \int_{-\infty}^{\infty} V(F) e^{j2\pi Ft}\, dF. \tag{2}$$

Other functions which do not satisfy the Dirichlet conditions are admissible if we allow the use of delta functions.

2.1 Dirac Delta Function

The Dirac delta (impulse function) can be defined in terms of its properties [1]. The sampling property of the delta function (more properly a distribution) is

$$\int_{t \in T_A} v(t)\delta(t)\, dt = \begin{cases} v(0) & 0 \in T_A \\ 0 & 0 \notin T_A. \end{cases} \tag{3}$$

From this characterization, the delta function can be shown to be zero everywhere except at the origin, yet it has unit area,

$$\begin{aligned} \delta(t) &= 0 \qquad \text{for } t \neq 0, \\ \int_{-\infty}^{\infty} \delta(t)\,dt &= 1. \end{aligned} \tag{4}$$

The formal operations involving the delta function in an integral give us

$$\int_{-\infty}^{\infty} v(t-t_o)\delta(t)\,dt = \int_{-\infty}^{\infty} v(t)\delta(t+t_o)\,dt \tag{5}$$

and

$$\int_{-\infty}^{\infty} v(at)\delta(t)\,dt = \frac{1}{|a|}\int_{-\infty}^{\infty} v(t)\delta(t/a)\,dt. \tag{6}$$

2.1.1 Fourier Transforms of Delta Functions

Using the sampling property of the delta function, the Fourier transform of the delta function evaluates to a constant,

$$\int_{-\infty}^{\infty} \delta(t)e^{-j2\pi Ft}\,dt = 1. \tag{7}$$

The inverse Fourier transform gives us

$$\int_{-\infty}^{\infty} e^{j2\pi Ft}\,dF = \delta(t). \tag{8}$$

This integral must be evaluated using the Cauchy principal value, i.e. as the limit

$$\lim_{T\to\infty}\int_{-T/2}^{T/2} e^{j2\pi Ft}\,dF. \tag{9}$$

Note that since $\delta(t)$ behaves like a symmetric function, the exponent in the integral can have either sign.

The inverse transform giving a delta function gives us a relation for the integral of a complex exponential. Here we restate the result using symbols which do not evoke time or frequency,

$$\boxed{\int_{-\infty}^{\infty} e^{\pm j2\pi ux}\,du = \delta(x).} \tag{10}$$

2.2 Fourier Series – Continuous-Time Signals

A periodic function (subject to conditions of absolutely integrability over a period, a finite number of finite discontinuities in a finite interval, and a finite number of extrema in a finite interval) has a Fourier series expansion in complex exponentials [2]. Consider a periodic function $\tilde{v}(t)$ with period T. The Fourier series expansion for $\tilde{v}(t)$ is

$$\tilde{v}(t) = \sum_{m=-\infty}^{\infty} v_m e^{j2\pi mt/T}. \tag{11}$$

The Fourier series coefficients are found from

$$v_m = \frac{1}{T} \int_{-T/2}^{T/2} \tilde{v}(t) e^{-j2\pi mt/T} \, dt. \tag{12}$$

2.3 Fourier Transform of a Continuous-Time Periodic Signal

We can now plunge ahead and express the Fourier transform of $\tilde{v}(t)$ in terms of the Fourier series coefficients,

$$\begin{aligned} V_p(F) &= \int_{-\infty}^{\infty} \tilde{v}(t) e^{-j2\pi Ft} \, dt \\ &= \sum_{m=-\infty}^{\infty} v_m \int_{-\infty}^{\infty} e^{-j2\pi t(F-m/T)} \, dt \\ &= \sum_{m=-\infty}^{\infty} v_m \delta\Big(F - \frac{m}{T}\Big). \end{aligned} \tag{13}$$

We have used Eq. (10) to evaluate the integral of the complex exponential. Summarizing, the Fourier transform of a periodic function is a sequence of delta functions in the frequency domain (at the harmonics of the periodic signal repetition rate). The areas of the delta functions are given by the Fourier series coefficients,

$$\boxed{V_p(F) = \sum_{m=-\infty}^{\infty} v_m \delta\Big(F - \frac{m}{T}\Big).} \tag{14}$$

2.4 Fourier Transform of a Periodic Impulse Train

Consider the periodic impulse train (period T),

$$\tilde{v}(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT). \tag{15}$$

The Fourier series coefficients for this signal are given by

$$v_m = \frac{1}{T} \int_{-T/2}^{T/2} \sum_{k=-\infty}^{\infty} \delta\Big(t - \frac{k}{T}\Big) e^{j2\pi mt/T} \, dt. \tag{16}$$

We see that the only delta function within the integration range is the one for $k = 0$. Using the sampling property of the delta function, the integral evaluates to unity. Then the Fourier series coefficients are constant ($v_m = 1/T$) and the Fourier transform of the impulse train is

$$V_p(F) = \frac{1}{T} \sum_{m=-\infty}^{\infty} \delta\Big(F - \frac{m}{T}\Big). \tag{17}$$

Periodic functions have delta functions in their Fourier transforms and delta functions have periodic functions in their Fourier transforms. Because of the duality between the forward and inverse Fourier transforms (if $v(t) \Longleftrightarrow V(F)$, then $V(t) \Longleftrightarrow v(-F)$), this gives us the result that an impulse train (periodic and delta functions) must have as its Fourier transform another impulse train (delta functions and periodic).

We have an alternate formulation for the Fourier transform (or inverse Fourier transform) of an impulse train. The Fourier transform for a delayed delta function $\delta(t - kT)$ is $e^{-j2\pi kTF}$. Then the Fourier transform pair is

$$\sum_{k=-\infty}^{\infty} \delta(t - kT) \Longleftrightarrow \sum_{k=-\infty}^{\infty} e^{-j2\pi kTF}. \tag{18}$$

An finally using the time-frequency duality of the Fourier transform, we get

$$\boxed{\sum_{k=-\infty}^{\infty} \delta(t - kT) = \frac{1}{T} \sum_{m=-\infty}^{\infty} e^{j2\pi mt/T} \Longleftrightarrow \sum_{k=-\infty}^{\infty} e^{-j2\pi kFT} = \frac{1}{T} \sum_{m=-\infty}^{\infty} \delta\Big(F - \frac{m}{T}\Big).} \tag{19}$$

2.5 Periodic Wrapped Continuous-Time Signals

Consider forming a periodic signal $\tilde{v}(t)$ from a (non-periodic) signal $v(t)$ as follows

$$\tilde{v}(t) = v(t) * \sum_{k=-\infty}^{\infty} \delta(t - kT) = \sum_{k=-\infty}^{\infty} v(t - kT). \tag{20}$$

We refer to this process which forms a time-aliased periodic signal as *wrapping*.[1] Using the fact that a convolution in the time-domain corresponds to a product in the frequency domain, the Fourier transform of $\tilde{v}(t)$ is

$$\sum_{k=-\infty}^{\infty} v(t - kT) \Longleftrightarrow V(F)\frac{1}{T} \sum_{m=-\infty}^{\infty} \delta\Big(F - \frac{m}{T}\Big) = \frac{1}{T} \sum_{m=-\infty}^{\infty} V\Big(\frac{m}{T}\Big)\delta\Big(F - \frac{m}{T}\Big). \tag{21}$$

[1]The continuous-time signal can be considered to be wrapped onto a circle of circumference T with all the superimposed intervals being added.

This shows that the Fourier series coefficients are just $V(F)/T$ evaluated at the harmonic frequencies. Here $V(F)$ is the Fourier transform of $v(t)$, where $v(t)$ can be longer than T.

We can also get the Fourier series coefficients directly from Eq. (12). We note that this equation is just a scaled version of the Fourier transform,

$$v_m = \frac{1}{T} V_T\Big(\frac{m}{T}\Big), \tag{22}$$

where $V_T(F)$ is the Fourier transform of one period of $\tilde{v}(t)$.

Given a signal $v(t)$ which is wrapped to become $\tilde{v}(t)$, there are two ways to get the coefficients of the frequency response of $\tilde{v}(t)$. The first is to take the Fourier transform of $v(t)$ (which can be longer than T) and then sample the frequency response at $F = m/T$. The second is to take the Fourier transform of one period of $\tilde{v}(t)$ and then sample the frequency response at $F = m/T$.

2.5.1 Poisson Sum Formula

From Eq. (21), one can take the term-by-term inverse Fourier transform of the extreme righthand side expression and equate it to the lefthand side. This gives the Poisson sum formula,

$$\boxed{\sum_{k=-\infty}^{\infty} v(t-kT) = \frac{1}{T} \sum_{m=-\infty}^{\infty} V\Big(\frac{m}{T}\Big) e^{j2\pi t m/T}.} \tag{23}$$

3 Discrete-Time Fourier Transform

The discrete-time Fourier transform (DTFT) is given by

$$V(\omega) = \sum_{n=-\infty}^{\infty} v[n] e^{-j\omega n}. \tag{24}$$

This sum converges if $v[n]$ is absolutely summable. The frequency response is periodic, with period 2π. This sum can be considered to be a Fourier series expansion of the periodic signal $V(\omega)$. The inverse discrete-time Fourier transform is the computation of the Fourier series coefficients,

$$v[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} V(\omega) e^{j\omega n}\, d\omega. \tag{25}$$

3.1 Fourier Transform of a Discrete-Time Periodic Signal

Let $\tilde{v}[n]$ be periodic with period N,

$$\tilde{v}[n+N] = \tilde{v}[n]. \tag{26}$$

Let us evaluate the Fourier transform of this signal.

$$\begin{aligned} V_p\omega) &= \sum_{n=-\infty}^{\infty} \tilde{v}[n]e^{-j\omega n} \\ &= \sum_{p=-\infty}^{\infty} \sum_{q=0}^{N-1} \tilde{v}[pN+q]e^{-j\omega(pN+q)} \\ &= \sum_{p=-\infty}^{\infty} e^{-j\omega pN} \sum_{q=0}^{N-1} \tilde{v}[q]e^{-j\omega q}. \end{aligned} \tag{27}$$

The second line of this equation is a result of substituting $n = pN + q$. The third line results from exploiting the periodicity of $\tilde{v}[n]$. The second factor of the result above is the DTFT of one period of $\tilde{v}[n]$. In the last line, the first factor (sum of complex exponentials) can be expressed in terms of an impulse train. The form of the sum is a little different than that encountered earlier. Appendix A recasts the earlier results in terms of radian frequency. Then from Eq. (63) with $T = N$,

$$\sum_{p=-\infty}^{\infty} e^{-j\omega pN} = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta\Big(\omega - \frac{2\pi k}{N}\Big). \tag{28}$$

Finally we can write

$$\boxed{V_p(\omega) = 2\pi \sum_{k=-\infty}^{\infty} V_k \delta\Big(\omega - \frac{2\pi k}{N}\Big),} \tag{29}$$

where

$$\boxed{V_k = \frac{1}{N} \sum_{n=0}^{N-1} \tilde{v}[n]e^{-j2\pi nk/N}.} \tag{30}$$

The coefficients V_k are periodic with period N. They are obtained as the DTFT of one period of $\tilde{v}[n]$, evaluated at $\omega = 2\pi k/N$. As we will see later, these coefficients are the same as the discrete Fourier transform, except for a scale factor.

3.2 Fourier Series – Discrete-Time Signals

The Fourier series expansion for a discrete time signal can be obtained by taking the inverse transform of Eq. (29),

$$\begin{aligned}\tilde{v}[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} V(\omega)e^{j\omega n}\,d\omega \\ &= \sum_{k=-\infty}^{\infty} V_k \int_{-\epsilon}^{2\pi-\epsilon} \delta\Big(\omega - \frac{2\pi k}{N}\Big)e^{j\omega n}\,d\omega \\ &= \sum_{k=0}^{N-1} V_k e^{j2\pi kn/N}.\end{aligned} \tag{31}$$

In the second line, the limits of the integration have been shifted so that the delta functions for $k = 0$ to $k = N - 1$ fall within the limits. This is possible because the integrand is periodic with period 2π. The result is a Fourier series expansion with Fourier series coefficients V_k,

$$\boxed{\tilde{v}[n] = \sum_{k=0}^{N-1} V_k e^{j2\pi kn/N}.} \tag{32}$$

3.3 Fourier Transform of a Discrete-Time Pulse Train

Consider the discrete-time pulse train

$$\tilde{v}[n] = \sum_{k=-\infty}^{\infty} \delta[n - kN]. \tag{33}$$

Here the delta function with square brackets is the unit pulse, equal to one if its argument is zero, and equal to zero otherwise. The Fourier series coefficients for this signal are constants at $1/N$, giving the Fourier series representation,

$$\sum_{k=-\infty}^{\infty} \delta[n - kN] = \frac{1}{N}\sum_{k=0}^{N-1} e^{j2\pi nk/N}. \tag{34}$$

The DTFT of this pulse train can be found term-by-term for the left-hand side of the equation above,

$$V_p(\omega) = \sum_{k=-\infty}^{\infty} e^{-j\omega kN}. \tag{35}$$

We can find a impulse train representation for this expression from Eq. (63) in Appendix A. This gives the following representations of a discrete-time pulse train.

$$\boxed{\sum_{k=-\infty}^{\infty} \delta[n-kN] = \frac{1}{N}\sum_{m=0}^{N-1} e^{j2\pi nm/N} \Longleftrightarrow \sum_{k=-\infty}^{\infty} e^{-j\omega kN} = \frac{2\pi}{N}\sum_{m=-\infty}^{\infty} \delta\Big(\omega - \frac{2\pi m}{N}\Big).} \tag{36}$$

This expression has discrete-time pulses on the left and delta functions on the right.

3.4 Periodic Wrapped Discrete-Time Signals

Consider forming a periodic signal $\tilde{v}[n]$ from $v[n]$,

$$\tilde{v}[n] = v[n] * \sum_{k=-\infty}^{\infty} \delta[n-kN] = \sum_{k=-\infty}^{\infty} v[n-kN]. \tag{37}$$

Using the fact that a convolution in the time-domain corresponds to a product in the frequency domain, the DTFT of $\tilde{v}[n]$ is

$$\sum_{k=-\infty}^{\infty} v[n-kN] \Longleftrightarrow V(\omega)\frac{2\pi}{N}\sum_{m=-\infty}^{\infty} \delta\Big(\omega - \frac{2\pi m}{N}\Big) = \frac{2\pi}{N}\sum_{m=-\infty}^{\infty} V\Big(\frac{2\pi m}{N}\Big)\delta\Big(\omega - \frac{2\pi m}{N}\Big). \tag{38}$$

Given a signal $v[n]$ which is wrapped to become $\tilde{v}[n]$, there are two ways to get the coefficients of the frequency response. The first is to take the Fourier transform of $v[n]$ and then sample the frequency response at $\omega = 2\pi m/N$. The second is to take the Fourier transform of one period of $\tilde{v}[n]$ and then sample the frequency response at $\omega = 2\pi m/N$.

3.5 Poisson Sum Formula

For discrete-time signals we can find a result similar to the Poisson sum formula for continuous-time signals. In this case, taking the term-by-term inverse Fourier transform of the extreme right-hand side of the equation above,

$$\boxed{\sum_{k=-\infty}^{\infty} v[n-kN] = \frac{1}{N}\sum_{m=-\infty}^{\infty} V\Big(\frac{2\pi m}{N}\Big) e^{j2\pi nm/N}.} \tag{39}$$

4 Discrete Fourier Transform

The discrete Fourier transform (DFT) for a finite length sequence $x[n]$ is [3]

$$V[k] = \sum_{n=0}^{N-1} v[n] e^{-j2\pi nk}. \tag{40}$$

The DFT coefficients $V[k]$ are periodic with period N. The inverse discrete Fourier transform is

$$v[n] = \frac{1}{N} \sum_{k=0}^{N-1} V[k] e^{j2\pi nk}. \tag{41}$$

In this equation, $v[n]$ becomes periodic with period N. With this view, we see that the DFT formula Eq. (40) can be considered to operate on a finite length signal, or alternately, can be considered to operate on one period of a periodic signal. In the latter interpretation, the DFT formula essentially calculates the Fourier series coefficients of the periodic signal.

The DFT formula differs from the Fourier series formula in Eq. (31) only in scale factor. The Fourier series coefficient V_k is related to the DFT coefficient as

$$V[k] = NV_k. \tag{42}$$

5 Relationships Between the Frequency-Domain Representations

The previous results put us in a good position to examine the relationships between the frequency representations of continuous-time signals, sampled signals, and periodic signals. Figure 1 shows a schematic form of the relationships. Consider the time-domain signals shown in that figure. On the left side of the diagram, the signal $x[n]$ is formed by sampling $x(t)$ with sampling interval T. The signal $\tilde{x}[n]$ is formed by wrapping $x[n]$ with period N. On the right side of the diagram, the signal $\tilde{x}(t)$ is formed by wrapping $x(t)$ with period NT. Sampling $\tilde{x}(t)$ with period T closes the loop and gives us $\tilde{x}[n]$. Thus sampling then wrapping (on the left side) is the same as wrapping and then sampling (on the right side – with the proviso that wrapping period is N times the sampling interval T).

5.1 Sampling a Continuous-Time Signal: $x(t) \rightarrow x[n]$

We can model the sampling of a continuous-time signal as the multiplication of the continuous-time signal by a impulse train. The areas of the resulting impulses are the sample values,

$$x_s(t) = x(t) \sum_{k=-\infty}^{\infty} \delta(t - kT) = \sum_{k=-\infty}^{\infty} x(kT)\delta(t - kT). \tag{43}$$

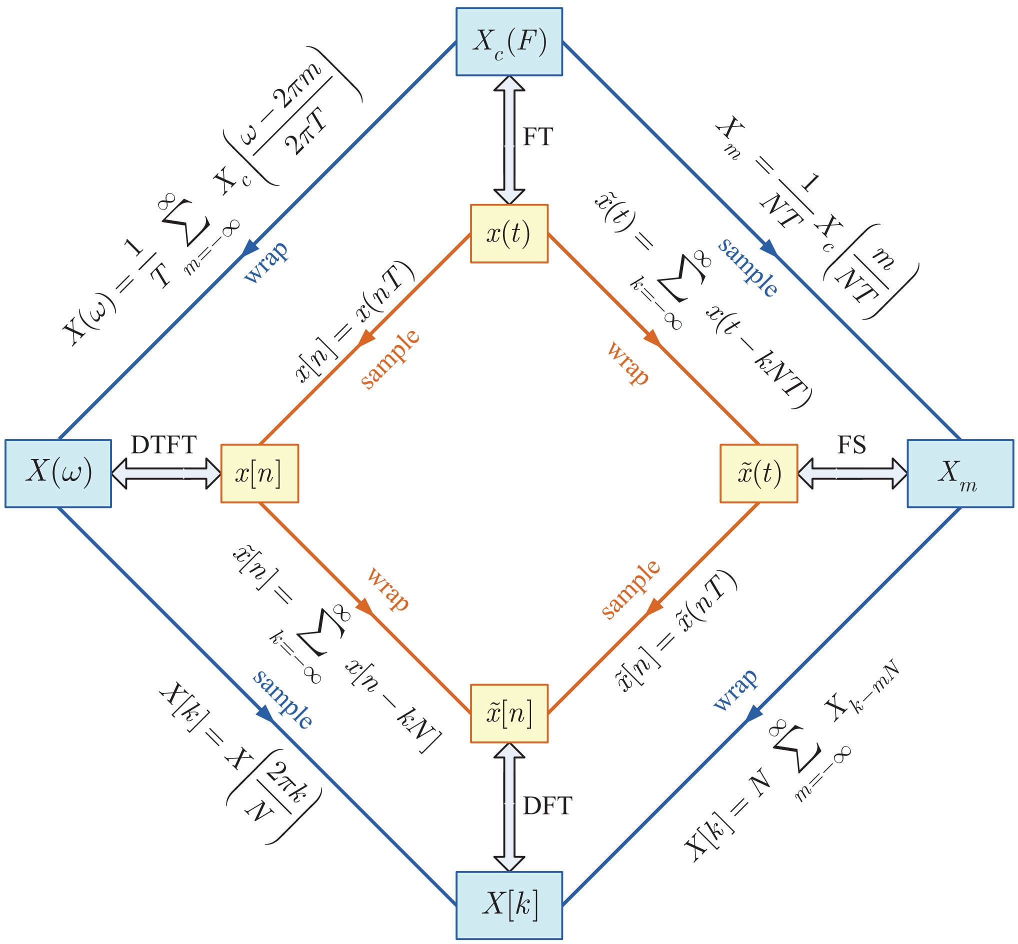

Fig. 1 Relationships between the frequency domain representations of continuous-time signals, sampled signals, and periodic signals. FT is the (continuous-time) Fourier transform, DTFT is the discrete-time Fourier transform, FS is the Fourier series, and DFT is the discrete Fourier transform.

The Fourier transform $X_s(F)$ can be computed using the relationship that a product in the time domain corresponds to a convolution in the frequency domain.

$$x_s(t) = x(t) \sum_{k=-\infty}^{\infty} \delta(t-kT) \Longleftrightarrow X_s(F) = X_c(F) * \frac{1}{T} \sum_{m=-\infty}^{\infty} \delta\Big(F - \frac{m}{T}\Big) = \frac{1}{T} \sum_{m=-\infty}^{\infty} X_c\Big(F - \frac{m}{T}\Big). \quad (44)$$

This frequency response is periodic with period $1/T$. In this form we see that sampling in the time domain corresponds to wrapping in the frequency domain. The resulting frequency response will not have aliasing (overlapping responses) if the baseband signal $X_c(F)$ is bandlimited to $|F| < 1/(2T)$.

There is another expression for $X_s(F)$. This time we take the Fourier transform term by term of the second form of the time domain expression in Eq. (43),

$$x_s(t) = \sum_{k=-\infty}^{\infty} x(kT)\delta(t-kT) \Longleftrightarrow X_s(F) = \sum_{k=-\infty}^{\infty} x(kT) e^{-j2\pi kFT}. \quad (45)$$

In discrete-time,

$$x[n] = x(nT). \quad (46)$$

The DTFT of $x[n]$ is $X(\omega)$ which is periodic with period 2π. If we compare the definition for the DTFT (Eq. (24)) with Eq. (45), we see that

$$X(\omega) = X_x(F)\big|_{F=\omega/(2\pi T)} = \sum_{k=-\infty}^{\infty} x[k] e^{-j\omega k} \quad (47)$$

The mapping between F for the continuous-time Fourier transform and ω for the discrete-time Fourier transform is $\omega = 2\pi FT$. With this mapping, when F increases by $1/T$, ω increases by 2π.

Returning to the wrapped form of the frequency response in Eq. (44), the DTFT can be written as,

$$\boxed{X(\omega) = \frac{1}{T} \sum_{m=-\infty}^{\infty} X_c\Big(\frac{\omega - 2\pi m}{2\pi T}\Big).} \quad (48)$$

As pointed out earlier, the $x[n] \Longleftrightarrow X(\omega)$ relationship is that of Fourier series coefficients $x[n]$ corresponding to a periodic signal $X(\omega)$.

5.2 Wrapping a Continuous-Time Signal: $x(t) \to \tilde{x}(t)$

The frequency-domain consequences of wrapping a continuous-time signal have been explored in Section 2.5. That result is reproduced here with the appropriate change of variables,

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} x(t - kNT) \Longleftrightarrow \frac{1}{NT} \sum_{m=-\infty}^{\infty} X_c\Big(\frac{m}{NT}\Big)\delta\Big(F - \frac{m}{NT}\Big). \tag{49}$$

In the diagram, the frequency domain representation of the wrapped sequence is given in terms of its continuous-time Fourier series coefficients,

$$\boxed{x_m = \frac{1}{NT} X_c\Big(\frac{m}{NT}\Big).} \tag{50}$$

5.3 Wrapping a Discrete-Time Signal: $x[n] \to \tilde{x}[n]$

The frequency-domain consequences of wrapping a discrete-time signal have been explored in Section 3.4. That result is reproduced here with the appropriate change of variables,

$$\tilde{x}[n] = \sum_{k=-\infty}^{\infty} x[n - kN] \Longleftrightarrow \frac{2\pi}{N} \sum_{m=-\infty}^{\infty} X\Big(\frac{2\pi m}{N}\Big)\delta\Big(\omega - \frac{2\pi m}{N}\Big). \tag{51}$$

The discrete-time Fourier series coefficients for $\tilde{x}[n]$ are

$$X_m = \frac{1}{N} X\Big(\frac{2\pi m}{N}\Big). \tag{52}$$

We can substitute for $X(\omega)$ from Eq. (48) to get an expression for the Fourier series coefficients directly in terms of wrapped samples of the continuous-time Fourier transform $X_c(F)$,

$$X_k = \frac{1}{NT} \sum_{m=-\infty}^{\infty} X_c\Big(\frac{k - mN}{NT}\Big). \tag{53}$$

In the figure, the corresponding relationship is expressed in terms of the discrete Fourier transform coefficients ($X[k] = NX_k$),

$$\boxed{X[k] = X\Big(\frac{2\pi k}{N}\Big).} \tag{54}$$

These coefficients are the DFT for one period of $\tilde{x}[n]$.

5.4 Sampling a Continuous-Time Periodic Signal: $\tilde{x}(t) \rightarrow \tilde{x}[n]$

The periodic signal $\tilde{x}(t)$ is represented by its Fourier series coefficients x_m in Eq. (50). The periodic discrete-time signal $\tilde{x}[n]$ is likewise represented by its Fourier series coefficients X_m in Eq. (53). The relationship between these is

$$X_k = \sum_{m=-\infty}^{\infty} x_{k-mN}. \tag{55}$$

Finally, the DFT coefficients expressed in terms of the Fourier series coefficients of $\tilde{x}(t)$ are given by

$$\boxed{X[k] = N \sum_{m=-\infty}^{\infty} x_{k-mN}.} \tag{56}$$

5.5 Frequency Domain Relationships

5.5.1 Reversibility

The diagram showing the frequency domain relationships shows that sampling in one domain corresponds to wrapping in the other domain. The diagram shows directed arrows for the sampling and wrapping operations. Under some circumstances, one can "reverse" the operation. For instance, sampling is reversible if a time signal is appropriately bandlimited. Similarly, wrapping a time signal is reversible if the signal is time limited to less than the wrapping period. However to reach the DFT from the Fourier transform involves both sampling and wrapping. The combination is not reversible since a signal cannot be simultaneously bandlimited and time limited.

5.5.2 Periodic $x(t)$

Consider a periodic continuous-time signal $x(t)$. Sampling this signal is well-defined. However, wrapping this signal can result in the sum becoming infinite. Since going from continuous-time to the DFT input $\tilde{x}[n]$ involves both sampling and wrapping, this is generally not possible for periodic $x(t)$.

Consider

$$x(t) = e^{j2\pi F_o t}. \tag{57}$$

This periodic signal a Fourier series with a single term and thus has a Fourier transform consisting of a single delta function at $F = F_o$. Sampling $x(t)$ at kT results in a DTFT which has a delta functions at $\omega = 2\pi F_o T + 2\pi m$. There is a single delta function in any interval of length 2π.

Now consider a more general periodic signal which has an unbounded number of harmonics,

$$x(t) = \sum_{m=-\infty}^{\infty} x_m e^{j2\pi m F_o t}. \tag{58}$$

If we sample this signal at kT, there are two cases. If $F_o T = M/N$ where M and N are relatively prime, then the discrete-time signal will be periodic with period N. Since the Fourier series expansion for a periodic signal with period N has at most N terms, the DTFT will contain at most N delta functions in every interval of length 2π. If FoT is irrational, the DTFT can potentially contain an infinite number of delta functions in every interval of length 2π.

The conclusion is that sampled periodic signals have a DTFT consisting of a finite number of delta function per 2π interval if the sampled signal is itself periodic (requires the sampling interval be synchronized with the period), or if the periodic signal has a finite number of harmonics (Fourier series expansion with a finite number of terms).

6 Summary

These notes has shown that the Fourier transform can be applied to periodic continuous-time or discrete-time signals. This allows for a unified analysis of signals containing both non-periodic and periodic components. The second part of these notes have examined the frequency domain relationships for signals derived by sampling and/or wrapping a continuous-time signal.

Appendix A Continuous-Time Results Expressed in Radian Measure

In this appendix, we restate the results derived earlier for continuous-time signals using radian frequency. Using radian frequency (Ω), the Fourier transform is

$$V(\Omega) = \int_{-\infty}^{\infty} v(t) e^{-j\Omega t} \, dt. \tag{59}$$

The inverse transform is

$$v(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} V(\Omega) e^{j\Omega t} \, d\Omega. \tag{60}$$

The integral representation for a delta function in Eq. (10) has a 2π factor in the exponent. Absorbing this factor into the variable u, a modified integral representation is

$$\boxed{\int_{-\infty}^{\infty} e^{\pm jux} \, du = 2\pi\delta(x).} \tag{61}$$

Using this result, the Fourier transform of a periodic sequence expressed in terms of Ω is (c.f. Eq. (14))

$$\boxed{V_p(\Omega) = 2\pi \sum_{m=-\infty}^{\infty} v_m \delta\Big(\Omega - \frac{2\pi m}{T}\Big),} \tag{62}$$

where v_m is given in Eq. (12). The Fourier transform of the periodic impulse train is (c.f. Eq. (19))

$$\boxed{\sum_{k=-\infty}^{\infty} \delta(t - kT) = \frac{1}{T} \sum_{m=-\infty}^{\infty} e^{j2\pi mt/T} \Longleftrightarrow \sum_{k=-\infty}^{\infty} e^{-jk\Omega T} = \frac{2\pi}{T} \sum_{m=-\infty}^{\infty} \delta\Big(\Omega - \frac{2\pi m}{T}\Big).} \tag{63}$$

The Poisson sum formula for the Fourier transform with radian argument is (c.f. Eq. (23))

$$\boxed{\sum_{k=-\infty}^{\infty} v(t - kT) = \frac{1}{T} \sum_{m=-\infty}^{\infty} V\Big(\frac{2\pi m}{T}\Big) e^{j2\pi tm/T}.} \tag{64}$$

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